

# 带有广义记忆核 Caputo 分数阶导数的一种新数值离散格式

胡小兰, 梁宗旗

(集美大学理学院, 福建 厦门 361021)

**[摘要]** 本文主要研究了带有广义记忆核 Caputo 型分数阶导数的  $L_1$  差分格式。利用  $L_1$  线性插值和降阶法构造了带有广义记忆核  $\alpha(1 < \alpha < 2)$  阶 Caputo 型分数阶导数的离散格式, 研究了其系数性质, 并给出了其截断误差, 收敛阶为  $O(\tau^{3-\alpha})$ 。最后, 通过数值算例验证了该格式的有效性和数值精度。

**[关键词]** Caputo 分数阶导数;  $L_1$  插值; 降阶法; 收敛阶

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## A New Numerical Discrete Scheme of Caputo Fractional Derivative with Generalized Memory Kernel

HU Xiaolan, LIANG Zongqi

(School of Science, Jimei University, Xiamen 361021, China)

**Abstract:** The  $L_1$  difference scheme of Caputo-fractional derivative with the generalized memory kernel is proposed in this paper. By the linear interpolation and reduced order method, the discrete numerical scheme of Caputo-fractional derivative with the generalized memory kernel  $\alpha(1 < \alpha < 2)$ , namely  $L_1$  numerical scheme, is presented, and the nature of the coefficients is also proved. Additionally, an error estimate of numerical scheme is provided, demonstrating that the convergence order is  $O(\tau^{3-\alpha})$ , where  $\tau$  and  $\alpha \in (1, 2)$  are the time step size and the fractional order, respectively. The numerical example is presented to demonstrate the effectiveness of the proposed method and the numerical accuracy is validated.

**Keywords:** Caputo fractional derivative;  $L_1$  interpolation; order reduction method; convergence order

## 0 引言

分数阶导数微分方程可用于精确地描述物理和化学的记忆<sup>[1-2]</sup>。记忆可用“记忆函数”来描述。由于大多数分数算子存在奇异核, 因此无法给出分数阶微分方程的解析解, 即便存在解析解, 其收敛速度也非常慢, 所以利用数值方法求解分数阶微分方程是一种自然的选择途径。关于分数阶微分方程的求解, 有很多种方法, 如有限元方法<sup>[3]</sup>、谱方法<sup>[7]</sup>、有限差分方法<sup>[5,8]</sup>等。其中有限差分方法计算

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**[作者简介]** 通信作者: 梁宗旗(1964—), 教授, 硕士生导师, 从事计算数学方向研究。E-mail: zqliang@jmu.edu.cn

简单、易行, 故得以被广泛使用。

常见的 Caputo 型分数阶导数的离散方法有  $L_1$ 、 $L_{2-1\sigma}$  和  $L_2$  离散。针对  $\alpha \in (0, 1)$ , 利用传统的  $L_1$  公式逼近 Caputo 分数阶导数, 其主要思想是用分段线性插值多项式替换积分内的被积函数, 达到  $O(\tau^{2-\alpha})$  的精度<sup>[5-6]</sup>。Alikhanov<sup>[9]</sup>利用分段二次插值多项式, 推出一个  $L_{2-1\sigma}$  公式, 用来近似 Caputo 导数, 其收敛阶为  $O(\tau^{3-\alpha})$ ; Alikhanov<sup>[10]</sup>等利用  $L_2$  插值对 Caputo 导数进行离散, 其收敛阶达到  $O(\tau^{3-\alpha})$ ; TrifceSandev<sup>[11]</sup>研究了具有广义记忆核的 Fokker-Planck-Smoluchowski 方程, 证明了出现在广义扩散方程中的记忆核具有多种形式, 可以描述广泛的实验现象; 闰羽媛<sup>[12]</sup>在研究无界域上薛定谔方程的人工边界问题时, 得到了一个带有广义记忆核的 Caputo 分数阶导数形式, 其广义函数为 Mittag-leffler 函数; Trifcesandev<sup>[13]</sup>研究了广义波方程, 推导了广义波方程的解, 证明了波方程存在不同形式的广义内存核。由于针对带有广义记忆核 Caputo 导数的研究相对罕见, 故对带有广义记忆核 Caputo 分数阶导数进行离散化处理, 便成为了一个具有研究价值的研究课题。Alikhanov 等<sup>[14]</sup>研究了具有广义记忆核的变系数次扩散方程, 对带有广义记忆核  $\alpha$  阶 Caputo 型分数阶导数构造了  $L_1$  离散格式, 达到了  $O(\tau^{2-\alpha})$  的精度, 并证明了差分格式是无条件稳定的和收敛的。Gu 等<sup>[15]</sup>研究了求解具有变系数的广义时空分数扩散方程的快速隐式差分格式。针对  $\alpha \in (1, 2)$ , Sun 等<sup>[16]</sup>研究了时间分数阶波方程的差分格式。本文利用降阶法, 研究了带有广义记忆核的  $\alpha(1 < \alpha < 2)$  阶 Caputo 型分数阶导数  $L_1$  差分格式, 分析了其系数性质, 证明了截断误差为  $O(\tau^{3-\alpha})$ , 并利用数值算例严格验证了格式的有效性。

带有广义记忆核  $\alpha(1 < \alpha < 2)$  阶 Caputo 分数阶导数<sup>[13-14]</sup>的形式如下

$$\partial_{at}^{\alpha, \lambda(t)} v(t) = (1/\Gamma(2-\alpha)) \int_a^t \lambda(t-\eta)(t-\eta)^{1-\alpha} v''(\eta) d\eta. \quad (1)$$

其中:  $v(t) \in L_2([0, T])$  且  $\lambda(t)$  称为广义记忆核。当  $\lambda(t) = 1$ , 式 (1) 退化为为一般传统 Caputo 分数阶导数。为方便计算, 本文中只讨论  $\alpha = 0$  的情形。

## 1 带有广义记忆核 Caputo 分数阶导数的 $L_1$ 数值离散格式

本节主要利用  $L_1$  插值对带有广义记忆核的  $\alpha(1 < \alpha < 2)$  阶 Caputo 分数阶导数进行数值逼近, 得到了其差分格式, 分析了其截断误差。

取正整数  $N, T$ , 记  $0 = t_0 < t_1 < \cdots < t_N = T$  是  $[0, T]$  上的一个剖分, 并记  $\tau = T/N$ ,  $t_n = n\tau$ ,  $1 \leq n \leq N$ , 则有  $\partial_{0t_{n+1}}^{\alpha, \lambda(t)} v(t) = (1/\Gamma(2-\alpha)) \int_0^{t_{n+1}} \lambda(t_{n+1}-\eta)(t_{n+1}-\eta)^{1-\alpha} v''(\eta) d\eta$ 。令  $g(t) = v'(t)$ , 在区间  $[t_s, t_{s+1}]$  对函数  $g(t)$  作  $L_1$  线性插值, 则有

$$\begin{aligned} \partial_{0t_{n+1}}^{\alpha, \lambda(t)} v(t) &= (1/\Gamma(2-\alpha)) \int_0^{t_{n+1}} \lambda(t_{n+1}-\eta)(t_{n+1}-\eta)^{1-\alpha} g'(\eta) d\eta = \\ &= (1/\Gamma(2-\alpha)) \sum_{s=0}^n \int_{t_s}^{t_{s+1}} \lambda(t_{n+1}-\eta)(t_{n+1}-\eta)^{1-\alpha} g'(\eta) d\eta = \\ &= (1/\Gamma(2-\alpha)) \left( \sum_{s=0}^n g_{t,s} \int_{t_s}^{t_{s+1}} \lambda(t_{n+1}-\eta)(t_{n+1}-\eta)^{1-\alpha} d\eta + \right. \\ &\quad \left. \sum_{s=0}^n \int_{t_s}^{t_{s+1}} \lambda(t_{n+1}-\eta)(g(\eta) - \Pi_{1,s}g(\eta))'(t_{n+1}-\eta)^{1-\alpha} d\eta \right) = \\ &= (1/\Gamma(2-\alpha)) \sum_{s=0}^n g_{t,s} \int_0^1 \lambda(t_{n+1}-z\tau-t_s)(t_{n+1}-z\tau-t_s)^{1-\alpha} \tau dz + R_1^{n+1}. \end{aligned} \quad (2)$$

其中:  $\lambda_s = \lambda(t_s)$ ;  $\Pi_{1,s}g(\eta) = g(t_{s+1})(\eta-t_s)/\tau + g(t_s)(t_{s+1}-\eta)\tau$ ;  $g_{t,s} = (g(t_{s+1}) - g(t_s))/\tau$ ,  $g(\eta) - \Pi_{1,s}g(\eta) = 0.5g''(\xi_s)(\eta-t_s)(\eta-t_{s+1})$ ,  $\eta \in [t_s, t_{s+1}]$ ,  $s = 0, 1, \cdots, n$ ;  $R_1^{n+1} = (1/\Gamma(2-\alpha)) \sum_{s=0}^n \int_{t_s}^{t_{s+1}} \lambda(t_{n+1}-\eta)(g(\eta) - \Pi_{1,s}g(\eta))'(t_{n+1}-\eta)^{1-\alpha} d\eta$ 。

式 (2) 右端第一项又可写为

$$\begin{aligned} & (1/\Gamma(2-\alpha)) \sum_{s=0}^n g_{t,s} \int_0^1 \lambda(t_{n+1}-z\tau-t_s)(t_{n+1}-z\tau-t_s)^{1-\alpha} \tau dz = (\tau^{2-\alpha}/\Gamma(2-\alpha)) \sum_{s=0}^n g_{t,s} \int_0^1 \lambda(t_{n-s+1-z})(n-s+1-z)^{1-\alpha} dz \\ & = (\tau^{2-\alpha}/\Gamma(2-\alpha)) \sum_{s=0}^n g_{t,s} \int_0^1 (\lambda_{n-s+1-z} - \lambda_{n-s+1/2} - (\lambda_{n-s} - \lambda_{n-s+1})(z-1/2))(n-s+1-z)^{1-\alpha} dz \\ & + (\tau^{2-\alpha}/\Gamma(2-\alpha)) \sum_{s=0}^n g_{t,s} \int_0^1 (\lambda_{n-s+1/2} + (\lambda_{n-s} - \lambda_{n-s+1})(z-1/2))(n-s+1-z)^{1-\alpha} dz = \\ & (\tau^{2-\alpha}/\Gamma(2-\alpha)) \sum_{s=0}^n g_{t,s} \int_0^1 (\lambda_{n-s+1/2} + (\lambda_{n-s} - \lambda_{n-s+1})(z-1/2))(n-s+1-z)^{1-\alpha} dz + R_2^{n+1}. \quad (3) \end{aligned}$$

$$\text{其中: } R_2^{n+1} = (\tau^{2-\alpha}/\Gamma(2-\alpha)) \sum_{s=0}^n g_{t,s} \int_0^1 (\lambda_{n-s+1-z} - \lambda_{n-s+1/2} - (\lambda_{n-s} - \lambda_{n-s+1})(z-1/2))(n-s+1-z)^{1-\alpha} dz.$$

令

$$\int_0^1 (l+1-z)^{1-\alpha} dz = a_l/(2-\alpha), a_l = (l+1)^{2-\alpha} - l^{2-\alpha}; \quad (4)$$

$$\int_0^1 (z-1/2)(l+1-z)^{1-\alpha} dz = b_l/(2-\alpha), b_l = [(l+1)^{3-\alpha} - l^{3-\alpha}]/(3-\alpha) - [(l+1)^{2-\alpha} + l^{2-\alpha}]/2. \quad (5)$$

将式 (3) 代入式 (2) 可得

$$\begin{aligned} \partial_{0t_{n+1}}^{\alpha, \lambda(t)} v(t) &= (\tau^{2-\alpha}/\Gamma(2-\alpha)) \sum_{s=0}^n g_{t,s} \int_0^1 (\lambda_{n-s+1/2} + (\lambda_{n-s} - \lambda_{n-s+1})(z-1/2))(n-s+1-z)^{1-\alpha} dz + R_1^{n+1} + \\ R_2^{n+1} &= (\tau^{2-\alpha}/\Gamma(3-\alpha)) \sum_{s=0}^n (\lambda_{n-s+1/2} a_{n-s} + (\lambda_{n-s} - \lambda_{n-s+1}) b_{n-s}) g_{t,s} + R_1^{n+1} + R_2^{n+1} = [c_0^{(\alpha)} g(t_{n+1}) - \sum_{s=1}^n (c_{n-s}^{(\alpha)} \\ &- c_{n-s+1}^{(\alpha)}) g(t_s) - c_n^{(\alpha)} g(t_0)] + R^{n+1}. \quad (6) \end{aligned}$$

其中:  $c_l^{(\alpha)} = \tau^{1-\alpha}(\lambda_{l+1/2} a_l + (\lambda_l - \lambda_{l+1}) b_l)/\Gamma(3-\alpha)$ ,  $R^{n+1} = R_1^{n+1} + R_2^{n+1}$ 。同理

$$\partial_{0t_n}^{\alpha, \lambda(t)} v(t) = [c_0^{(\alpha)} g(t_n) - \sum_{s=1}^{n-1} (c_{n-s-1}^{(\alpha)} - c_{n-s}^{(\alpha)}) g(t_s) - c_n^{(\alpha)} g(t_0)] + R^n. \quad (7)$$

将式 (6) 和式 (7) 相加, 并除以 2, 得到  $[\partial_{0t_{n+1}}^{\alpha, \lambda(t)} v(t) + \partial_{0t_n}^{\alpha, \lambda(t)} v(t)]/2 = [c_0^{(\alpha)} (g(t_{n+1}) + g(t_n))/2 -$

$$\begin{aligned} & \sum_{s=1}^n (c_{n-s}^{(\alpha)} - c_{n-s+1}^{(\alpha)}) (g(t_{s+1}) + g(t_s))/2 - c_n^{(\alpha)} g(t_0)] + [R^{n+1} + R^n]/2. \text{ 因为} \\ & (g(t_{s+1}) + g(t_s))/2 = (v'(t_{s+1}) + v'(t_s))/2 = (v(t_{s+1}) + v(t_s))/\tau + \tau^2 v'''(\xi_{s+1})/12, \xi_{s+1} \in (t_s, t_{s+1}), \quad (8) \end{aligned}$$

记

$$\delta_t v^{s+1/2} = (v(t_{s+1}) - v(t_s))/\tau, \quad (9)$$

将式 (8) 和式 (9) 式代入式 (10) 可得

$$\begin{aligned} & [\partial_{0t_n}^{\alpha, \lambda(t)} v(t) + \partial_{0t_{n+1}}^{\alpha, \lambda(t)} v(t)]/2 = [c_0^{(\alpha)} \delta_t v^{n+1/2} - \sum_{s=1}^n (c_{n-s}^{(\alpha)} - c_{n-s+1}^{(\alpha)}) \delta_t v^{s+1/2} - c_n^{(\alpha)} v'(t_0)] + \hat{R}^{n+1/2} := \\ & D_t^{\alpha, \lambda(t)} v(t_{n+1/2}) + \hat{R}^{n+1/2}. \quad (10) \end{aligned}$$

其中:  $D_t^{\alpha, \lambda(t)} v(t_{n+1/2}) = c_0^{(\alpha)} \delta_t v^{n+1/2} - \sum_{s=1}^n (c_{n-s}^{(\alpha)} - c_{n-s+1}^{(\alpha)}) \delta_t v^{s+1/2} - c_n^{(\alpha)} v'(t_0)$ ;

$$\hat{R}^{n+1/2} = [c_0^{(\alpha)} \tau^2 v'''(\xi_{n+1})/12 - \sum_{s=1}^n (c_{n-s}^{(\alpha)} - c_{n-s+1}^{(\alpha)}) \tau^2 v'''(\xi_s)/12] + [R^{n+1} + R^n]/2. \quad (11)$$

**定理 1** 设  $\alpha \in (1, 2)$ ,  $v(t) \in C^3[0, t_{n+1}]$ ,  $\lambda(t) \in C^2[0, t_{n+1}]$ 。当  $\lambda(t) > 0$ ,  $\lambda'(t) \leq 0$  时, 有  $|\hat{R}^{n+1/2}| = |[\partial_{0t_{n+1}}^{\alpha, \lambda(t)} v(t) + \partial_{0t_n}^{\alpha, \lambda(t)} v(t)]/2 - D_t^{\alpha, \lambda(t)} v(t_{n+1/2})| \leq (1/6\Gamma(3-\alpha) + 1/8\Gamma(2-\alpha) + 1/\Gamma(3-\alpha))\lambda(0) \max_{0 \leq t \leq t_{n+1}} |v'''(t)| \tau^{3-\alpha} + ((1/8\Gamma(3-\alpha) \max_{0 \leq t \leq t_{n+1}} |\lambda'(t)| + (1/4\Gamma(3-\alpha)) \max_{0 \leq t \leq t_{n+1}} |v''(t)| \max_{0 \leq t \leq t_{n+1}} |\lambda''(t)|) \tau^{2-\alpha} \tau^2 = O(\tau^{3-\alpha})$ 。

**证明** 由式 (11) 可知  $|\hat{R}^{n+1/2}| = |[\sum_{s=1}^n (c_{n-s}^{(\alpha)} - c_{n-s+1}^{(\alpha)}) \tau^2 v'''(\xi_s)/12] + [R^{n+1} + R^n]/2|$ 。对上式右端第一项进行估计  $c_0^{(\alpha)} \tau^2 v'''(\xi_{n+1})/12 - \sum_{s=1}^n (c_{n-s}^{(\alpha)} - c_{n-s+1}^{(\alpha)}) \tau^2 v'''(\xi_s)/12 \leq c_0^{(\alpha)} \tau^2 v'''(\xi_{n+1})/12 - (c_0^{(\alpha)} \tau^2 v'''(\xi_n)/12 - c_n^{(\alpha)} \tau^2 v'''(\xi_1)/12) \leq c_0^{(\alpha)} \max_{0 \leq t \leq t_{n+1}} |v'''(t)| \tau^2/6 \leq (\lambda(0)/6\Gamma(3-\alpha)) \max_{0 \leq t \leq t_{n+1}} |v'''(t)| \tau^{3-\alpha}$ 。接下来对截断误差  $|R_1^{n+1}|$ 、 $|R_2^{n+1}|$  分别估计,  $|R_1^{n+1}| = (1/\Gamma(2-\alpha)) \left| \sum_{s=0}^n \int_{t_s}^{t_{s+1}} \lambda(t_{n+1}-\eta)(g(\eta) - \Pi_{1,s}g(\eta))'(t_{n+1}-\eta)^{1-\alpha} d\eta \right| \leq (1/\Gamma(2-\alpha)) \left| \sum_{s=0}^{n-1} \int_{t_s}^{t_{s+1}} \lambda(t_{n+1}-\eta)(g(\eta) - \Pi_{1,s}g(\eta))'(t_{n+1}-\eta)^{1-\alpha} d\eta \right| + (1/\Gamma(2-\alpha)) \int_{t_n}^{t_{n+1}} \lambda(t_{n+1}-\eta) |g'(\eta) - g_{t,n}| (t_{n+1}-\eta)^{1-\alpha} d\eta \leq (1/\Gamma(2-\alpha)) \sum_{s=0}^{n-1} \int_{t_s}^{t_{s+1}} |(g(\eta) - \Pi_{1,s}g(\eta))'| (t_{n+1}-\eta)^{1-\alpha} d\eta + ((\alpha-1)/\Gamma(2-\alpha)) \sum_{s=0}^{n-1} \int_{t_s}^{t_{s+1}} |(g(\eta) - \Pi_{1,s}g(\eta))| \lambda(t_{n+1}-\eta) (t_{n+1}-\eta)^{-\alpha} d\eta + (1/\Gamma(2-\alpha)) \int_{t_n}^{t_{n+1}} \lambda(t_{n+1}-\eta) |g'(\eta) - g_{t,n}| (t_{n+1}-\eta)^{1-\alpha} d\eta \leq (\tau^2/8\Gamma(2-\alpha)) \max_{0 \leq t \leq t_{n+1}} |v'''(t)| \max_{0 \leq t \leq t_{n+1}} |\lambda'(t)| \sum_{s=0}^{n-1} \int_{t_s}^{t_{s+1}} (t_{n+1}-\eta)^{-\alpha+1} d\eta + ((\alpha-1)\lambda(0)\tau^2/8\Gamma(2-\alpha)) \max_{0 \leq t \leq t_{n+1}} |v'''(t)| \sum_{s=0}^{n-1} \int_{t_s}^{t_{s+1}} (t_{n+1}-\eta)^{-\alpha} d\eta + (\lambda(0)\tau/\Gamma(2-\alpha)) \max_{0 \leq t \leq t_{n+1}} |v'''(t)| \int_{t_n}^{t_{n+1}} (t_{n+1}-\eta)^{1-\alpha} d\eta \leq (T^{2-\alpha}\tau^2/8\Gamma(3-\alpha)) \max_{0 \leq t \leq t_{n+1}} |\lambda'(t)| \max_{0 \leq t \leq t_{n+1}} |v'''(t)| + (1/8\Gamma(2-\alpha) + 1/\Gamma(3-\alpha)) \lambda(0) \tau^{3-\alpha} \max_{0 \leq t \leq t_{n+1}} |v'''(t)|。$

对误差  $R_2^{n+1}$  进行估计,

$$|R_2^{n+1}| = (\tau^{2-\alpha}/\Gamma(2-\alpha)) \sum_{s=0}^n |g_{t,s}| \left| \int_0^1 (\lambda_{n-s+1-z} - \lambda_{n-s+1/2} - (\lambda_{n-s} - \lambda_{n-s+1})(z-1/2)) (n-s+1-z)^{1-\alpha} dz \right|. \quad (12)$$

将函数  $\lambda(t_{n-s+1-z})$ ,  $\lambda(t_{n-s+1})$ ,  $\lambda(t_{n-s})$  分别在  $t_{n-s+1/2}$  处泰勒展开

$$\lambda(t_{n-s+1-z}) = \lambda(t_{n-s+1/2}) + \lambda'(t_{n-s+1/2})(1/2-z)\tau + \lambda''(t_{n-s+1/2})(1/2-z)^2\tau^2/2 + O(\tau^3), \quad (13)$$

$$\lambda(t_{n-s+1}) = \lambda(t_{n-s+1/2}) + \lambda'(t_{n-s+1/2})\tau/2 + \lambda''(t_{n-s+1/2})(\tau/2)^2/2 + O(\tau^3), \quad (14)$$

$$\lambda(t_{n-s}) = \lambda(t_{n-s+1}) - \lambda'(t_{n-s+1})\tau/2 + \lambda''(t_{n-s+1/2})(\tau/2)^2/2 + O(\tau^3). \quad (15)$$

将式(13)~(15)代入式 (12) 得  $|R_2^{n+1}| = (\tau^{2-\alpha}/\Gamma(2-\alpha)) \sum_{s=0}^n |g_{t,s}| \left| \int_0^1 \lambda''(t_{n-s+1/2})((1/2-z)\tau)^2(n-s+1-z)^{1-\alpha}/2 dz \right| \leq (\tau^3/4\Gamma(2-\alpha)) \max_{0 \leq t \leq t_{n+1}} |g'(t)| \max_{0 \leq t \leq t_{n+1}} |\lambda''(t)| \sum_{s=0}^n \int_0^1 (t_{n-s+1}-z\tau)^{1-\alpha} dz \leq (\tau^2/4\Gamma(2-\alpha)) \max_{0 \leq t \leq t_{n+1}} |v''(t)| \max_{0 \leq t \leq t_{n+1}} |\lambda''(t)| \sum_{s=0}^n \int_{t_s}^{t_{s+1}} (t_{n+1}-\eta)^{1-\alpha} dz = (\tau^2/4\Gamma(2-\alpha)) \max_{0 \leq t \leq t_{n+1}} |v''(t)| \max_{0 \leq t \leq t_{n+1}} |\lambda''(t)| \int_0^{t_{n+1}} (t_{n+1}-\eta)^{1-\alpha} d\eta = (T^{2-\alpha}\tau^2/4\Gamma(3-\alpha)) \max_{0 \leq t \leq t_{n+1}} |v''(t)| \max_{0 \leq t \leq t_{n+1}} |\lambda''(t)|。所以  $|R^{n+1}| = |R_1^{n+1} + R_2^{n+1}| \leq (1/8\Gamma(2-\alpha) + 1/\Gamma(3-\alpha)) \lambda(0) \max_{0 \leq t \leq t_{n+1}} |v'''(t)| \tau^{3-\alpha} + ((1/8\Gamma(3-\alpha)) \max_{0 \leq t \leq t_{n+1}} |\lambda'(t)| + (1/4\Gamma(3-\alpha)) \max_{0 \leq t \leq t_{n+1}} |v''(t)| \max_{0 \leq t \leq t_{n+1}} |\lambda''(t)|) T^{2-\alpha} \tau^2。从而  $|\hat{R}^{n+1/2}| \leq (1/6\Gamma(3-\alpha) + 1/8\Gamma(2-\alpha) + 1/\Gamma(3-\alpha)) \lambda(0) \max_{0 \leq t \leq t_{n+1}} |v'''(t)| \tau^{3-\alpha} + ((1/8\Gamma(3-\alpha)) \max_{0 \leq t \leq t_{n+1}} |\lambda'(t)| + (1/4\Gamma(3-\alpha)) \max_{0 \leq t \leq t_{n+1}} |v''(t)| \max_{0 \leq t \leq t_{n+1}} |\lambda''(t)|) T^{2-\alpha} \tau^2 = O(\tau^{3-\alpha})。$$$

定理证毕。下面研究系数的性质。

**引理 1** 系数  $a_l$  是由式 (4) 定义的, 系数  $b_l$  是由式 (5) 定义的, 对任意的  $l=0, 1, \dots$ , 当  $\alpha \in (1, 2)$  时, 有  $a_0 > a_1 > \dots > a_l > 0, b_0 > b_1 > \dots > b_l > 0$ 。

**证明** 令  $a(x) = (x+1)^{2-\alpha} - x^{2-\alpha}, (x \geq 0)$ 。不难验证  $a'(x) < 0$ , 即系数  $a_l$  为单调递减的。由

于  $b_l = [(l+1)^{3-\alpha} - l^{3-\alpha}]/(3-\alpha) - [(l+1)^{2-\alpha} + l^{2-\alpha}]/2$ , 故系数  $b_l$  又可表示成函数积分形式, 即由梯形求积公式可得出  $b_l = \int_l^{l+1} x^{2-\alpha} dx - [(l+1)^{2-\alpha} + l^{2-\alpha}]/2 = - (l^{2-\alpha})' / 2 \big|_{x=\xi_l} = (2-\alpha)(\alpha-1)\xi_l^{-\alpha}/12$ 。其中:  $l < \xi_l < l+1$ 。所以当  $l \geq 0$  时,  $b_l > 0$ 。

令  $b(x) = (2-\alpha)(\alpha-1)x^{-\alpha}/12$ , 则  $b'(x) = (2-\alpha)(\alpha-1)(-\alpha)x^{-(\alpha+1)}/12$ 。当  $x > 0$  时, 可验证函数  $b'(x) < 0$ , 故  $b(x)$  是单调递减的。引理证毕。

**引理 2** 对任意的  $\alpha$  ( $1 < \alpha < 2$ ), 当  $b_l > 0$ ,  $l \geq 1$  时, 有以下等式成立  $b_l = [(l+1)^{2-\alpha} - l^{2-\alpha}]/(K_l - 1/2) = a_l(K_l - 1/2)$ 。其中:  $K_l = ((l+1)^{3-\alpha} - l^{3-\alpha} - (3-\alpha)l^{2-\alpha})/(3-\alpha)[(l+1)^{2-\alpha} - l^{2-\alpha}]$ , 且  $1/2 < K_l < 1/(3-\alpha)$ 。

**证明** 定义函数  $f_\alpha(x) = ((x+1)^{3-\alpha} - x^{3-\alpha} - (3-\alpha)x^{2-\alpha})((3-\alpha)[(x+1)^{2-\alpha} - x^{2-\alpha}])^{-1} = \int_0^1 ((z+x)^{2-\alpha} - x^{2-\alpha})((x+1)^{2-\alpha} - x^{2-\alpha})^{-1} dx$ ,  $x > 0$  和  $h_\alpha(z, x) = ((z+x)^{2-\alpha} - x^{2-\alpha})((x+1)^{2-\alpha} - x^{2-\alpha})^{-1} = (z \int_0^1 (x+\xi)^{1-\alpha} d\xi) (\int_0^1 (x+\xi)^{1-\alpha} d\xi)^{-1}$ ,  $0 < z < 1, x > 0$ 。显然  $f_\alpha(x) = \int_0^1 h_\alpha(z, x) dz$ , 对任意的  $x > 0, 0 < z < 1$ , 有不等式  $\int_0^1 (x+\xi)^{1-\alpha} d\xi < \int_0^1 (x+z\xi)^{1-\alpha} d\xi < \int_0^1 (zx+z\xi)^{1-\alpha} d\xi = z^{1-\alpha} \int_0^1 (x+\xi)^{1-\alpha} d\xi$  成立。函数  $h_\alpha(z, x)$  有  $z < h_\alpha(z, x) < z^{2-\alpha}$ 。对上述不等式中的  $z$  关于 0 到 1 积分, 有  $1/2 < f_\alpha(x) = \int_0^1 h_\alpha(z, x) dz < 1/(3-\alpha)$ , 即  $1/2 < K_l < 1/(3-\alpha)$ 。引理证毕。

**引理 3** 系数  $a_l$  是由式 (4) 定义的, 系数  $b_l$  是由式 (5) 定义的, 当  $\alpha \in (1, 2)$ , 对任意的  $l = 1, 2, 3, \dots$ , 有

$$(2-\alpha)(l+1)^{1-\alpha} < a_l < (2-\alpha)l^{1-\alpha}, \quad (16)$$

$$(\alpha-1)(2-\alpha)(l+2)^{-\alpha} < a_l - a_{l+1} < (\alpha-1)(2-\alpha)l^{-\alpha}, \quad (17)$$

$$(\alpha-1)(2-\alpha)(l+2)^{-\alpha}/12 < b_l < (\alpha-1)(2-\alpha)l^{-\alpha}/12. \quad (18)$$

**证明**  $a_l, a_l - a_{l+1}, b_l$  可表示成如下积分形式, 即

$$a_l = (2-\alpha) \int_0^1 (l+\xi)^{1-\alpha} d\xi, \quad (19)$$

$$a_l - a_{l+1} = (\alpha-1)(2-\alpha) \int_0^1 d\eta \int_0^1 (l+\xi+\eta)^{-\alpha} d\xi, \quad (20)$$

$$b_l = 2^{\alpha-3}(\alpha-1)(2-\alpha) \int_0^1 \eta d\eta \int_{2s+1-\eta}^{2s+1+\eta} \xi^{-\alpha} d\xi. \text{ 由式 (19) 可知, 式 (16) 显然成立。分析式 (20),}$$

根据积分中值定理有  $\int_0^1 (l+\xi+\eta)^{-\alpha} d\xi = (l+\eta+\zeta)^{-\alpha}, \zeta \in (0, 1)$ 。因为  $\eta \in (0, 1)$ , 故  $(l+2)^{-\alpha} < \int_0^1 d\eta \int_0^1 (l+\xi+\eta)^{-\alpha} d\xi = \int_0^1 (l+\eta+\zeta)^{-\alpha} d\eta < l^{-\alpha}$ , 故  $(\alpha-1)(2-\alpha)(l+2)^{-\alpha} < (\alpha-1)(2-\alpha) \int_0^1 d\eta \int_0^1 (l+\xi+\eta)^{-\alpha} d\xi < (\alpha-1)(2-\alpha)l^{-\alpha}$ , 即式 (17) 得证。同理可证式 (18) 成立。

**引理 4** 对任意的  $l=0, 1, \dots, \alpha \in (1, 2)$ , 并且  $\lambda(t) \in C^2[0, t_{n+1}]$ , 当  $\lambda(t) > 0, \lambda'(t) \leq 0$ , 时, 对任意  $t \in [0, T]$ , 有  $c_0^{(\alpha)} > c_1^{(\alpha)} > \dots > c_l^{(\alpha)} > \lambda(t_{l+1/2})t_{l+1}^{1-\alpha}/\Gamma(2-\alpha)$ 。

**证明** 当  $\lambda(t) > 0, \lambda'(t) \leq 0$  时,  $b_l$  恒正, 故  $c_l^{(\alpha)} = (\tau^{1-\alpha}/\Gamma(3-\alpha))(\lambda_{l+1/2}a_l + (\lambda_l - \lambda_{l+1})b_l) > \tau^{1-\alpha}\lambda_{l+1/2}a_l/\Gamma(3-\alpha)$ 。又由引理 3, 可得  $c_l^{(\alpha)} > \lambda(t_{l+1/2})t_{l+1}^{1-\alpha}/\Gamma(2-\alpha)$ 。

接下来证明系数  $c_l$  为单调递减的。

$$c_l - c_{l+1} = (\tau^{1-\alpha}/\Gamma(3-\alpha))(\lambda_{l+1/2}a_l + (\lambda_l - \lambda_{l+1})b_l - \lambda_{l+3/2}a_{l+1} - (\lambda_{l+1} - \lambda_{l+2})b_{l+1}) = (\tau^{1-\alpha}/\Gamma(3-\alpha))(\lambda_{l+1/2}a_l - \lambda_{l+3/2}a_{l+1} - \lambda_{l+1}b_l - \lambda_{l+1}b_{l+1} + \lambda_l b_l + \lambda_{l+2}b_{l+1}) >$$

$$\begin{aligned} &(\tau^{1-\alpha}/\Gamma(3-\alpha))(\lambda_{l+1/2}(a_l-a_{l+1}-b_l-b_{l+1})+\lambda_lb_l+\lambda_{l+2}b_{l+1})> \\ &(\tau^{1-\alpha}\lambda_{l+1/2}/\Gamma(3-\alpha))(a_l-a_{l+1}-b_l-b_{l+1}). \end{aligned}\tag{21}$$

当  $l=0$  时, 有  $c_0-c_1>(\tau^{1-\alpha}\lambda_{1/2}/\Gamma(3-\alpha))(a_0-a_1-b_0-b_1)=(\tau^{1-\alpha}\lambda_{1/2}/\Gamma(3-\alpha))(1-2^{2-\alpha}+1-(3-\alpha)^{-1}+1/2-2^{3-\alpha}(3-\alpha)^{-1}+(3-\alpha)^{-1}+2^{1-\alpha}+1/2)=(\tau^{1-\alpha}\lambda_{1/2}/\Gamma(3-\alpha))(3-2^{1-\alpha}-2^{3-\alpha}(3-\alpha)^{-1})=(\tau^{1-\alpha}\lambda_{1/2}/\Gamma(3-\alpha))(3-2^{1-\alpha}(1+4(3-\alpha)^{-1}))>0$ 。

当  $l\geq 1$  时, 根据引理 3, 式 (21) 可表示成  $c_l-c_{l+1}>(\tau^{1-\alpha}\lambda_{l+1/2}/\Gamma(3-\alpha))((\alpha-1)(2-\alpha)(l+2)^{-\alpha}-(\alpha-1)(2-\alpha)l^{-\alpha}/12-(\alpha-1)(2-\alpha)(l+1)^{-\alpha}/12)=((\alpha-1)(2-\alpha)(l+2)^{-\alpha}\lambda_{l+1/2}\tau^{1-\alpha}/12\Gamma(3-\alpha))(12-(l+2)^{\alpha}l^{-\alpha}-(l+2)^{\alpha}(l+1)^{-\alpha})=((\alpha-1)(2-\alpha)(l+2)^{-\alpha}\lambda_{l+1/2}\tau^{1-\alpha}/12\Gamma(3-\alpha))(12-((1+2l^{-1})^{\alpha}+(1+(l+1)^{-1})^{\alpha}))$ , 故当  $l=1$  时,  $3^{\alpha}+(3/2)^{\alpha}<12$ , 此时  $c_1-c_2>0$  显然成立。

当  $l\geq 2$  时, 令  $p(x)=(1+2x^{-1})^{\alpha}+(1+(x+1)^{-1})^{\alpha}, x\geq 2$ 。显然, 函数  $p'(x)<0$ , 即  $p(x)\leq p(2)=2^{\alpha}+(4/3)^{\alpha}<12$ , 即可证  $c_l-c_{l+1}>0$ 。引理证毕。

2 数值例子

本节将通过数值算例来验证差分格式 (10) 的有效性和数值精度。  
取正整数  $N$ , 记  $T=1, \Delta t=T/N=1/N, V^{n+1/2}=(\partial_{0t}^{\alpha,\lambda(t)}v(t_{n+1})+\partial_{0t}^{\alpha,\lambda(t)}v(t_n))/2$  且  $v^{n+1/2}(0\leq n\leq N)$  为  $D_t^{\alpha,\lambda(t)}$  的解, 记  $E_{\infty}^{n+1/2}(\Delta t)=\max_{0\leq n\leq N}|V^{n+1/2}-v^{n+1/2}|$ 。则在  $L_{\infty}$  意义下时间收敛阶为  $\text{Order}_{\infty}=\log_2(E_{\infty}^{n+1/2}(\Delta t)/(E_{\infty}^{n+1/2}(\Delta t/2)))$ 。其中:  $\Delta t$  为时间步长。

**例** 假设  $1<\alpha<2$ , 令  $v(t)=e^{-t}, \lambda(t)=e^{-t}$ , 计算  $\alpha(1<\alpha<2)$  阶带有广义记忆核 Caputo 分数阶导数  ${}_0^CD_t^{\alpha,\lambda}v(t)$  在  $T=1$  处的数值解。精确解为  ${}_0^CD_t^{\alpha,\lambda}v(t)|_{t=1}=e^{-t}t^{2-\alpha}/\Gamma(3-\alpha)|_{t=1}=e^{-1}/\Gamma(3-\alpha)$ 。取不同步长  $\Delta t=1/10, 1/20, 1/40, 1/80, 1/160$ 。表 1 列出了  $\alpha$  分别取 1.1, 1.5, 1.9 时, 不同步长下的最大模误差估计及收敛精度。由表 1 可知, 本文所构造的带有广义记忆核  $\alpha(1<\alpha<2)$  阶 Caputo 型时间分数阶导数的差分格式是一种简单, 易于实现的格式, 其收敛阶无限接近于  $O(\tau^{3-\alpha})$ 。

表 1 不同时间步长下的最大误差及收敛阶  
Tab.1 The maximum errors and convergence orders with different time step

$\Delta t$	$\alpha=1.1$		$\alpha=1.5$		$\alpha=1.9$	
	$E_{\infty}^{n+1/2}(\Delta t)$	$\text{Order}_t$	$E_{\infty}^{n+1/2}(\Delta t)$	$\text{Order}_t$	$E_{\infty}^{n+1/2}(\Delta t)$	$\text{Order}_t$
1/10	$2.1000\times 10^{-3}$	-	$0.9000\times 10^{-2}$	-	$2.5600\times 10^{-2}$	-
1/20	$5.8019\times 10^{-4}$	1.85	$3.2000\times 10^{-3}$	1.49	$1.2000\times 10^{-2}$	1.09
1/40	$1.5701\times 10^{-4}$	1.88	$1.2000\times 10^{-3}$	1.41	$5.6000\times 10^{-3}$	1.09
1/80	$4.2437\times 10^{-5}$	1.88	$4.1195\times 10^{-4}$	1.54	$2.6000\times 10^{-3}$	1.10
1/160	$1.1460\times 10^{-5}$	1.88	$1.4629\times 10^{-4}$	1.49	$1.2000\times 10^{-3}$	1.07

3 结论

在带有广义记忆核  $\alpha(0<\alpha<1)$  阶 Caputo 分数阶导数研究的基础上, 利用降阶法, 研究了带有广义记忆核的  $\alpha(1<\alpha<2)$  阶 Caputo 型分数阶导数  $L_1$  差分格式, 分析了其系数性质, 证明了截断误差为  $O(\tau^{3-\alpha})$ , 并通过数值算例严格验证了格式的有效性。本文提供的是带有广义记忆核的  $\alpha(1<\alpha<2)$  阶 Caputo 型分数阶导数的一种离散格式, 下一步的研究可以将这种差分格式应用到求解其他含有广义记忆核的 Caputo 分数阶导数的差分方程并进行误差估计, 同时也可以考虑研究带有广义记忆核  $\alpha(1<\alpha<2)$  阶 Caputo 分数阶导数的高阶差分格式, 进一步提高收敛精度。

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